MIDTERM REPORT: THE BILLIARD PROBLEM

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1. Background

For my midterm project, I decided to study the billiards problem. Overall, it interested me most because the behavior of a billiard ball can demonstrate chaotic behavior; even with seemingly simple assumptions like mirror-like collisions, the billiard's motion is complex and depends highly on the initial conditions.

The problem begins by considering a billiard ball placed at some initial position (x_0, y_0) on a table. We can consider tables of different sizes and dimensions, for example rectangles, circles, and other "stadium" configurations. We impart the billiard ball with some initial velocity $\vec{v}$, then follow its motion given by the following differential equations:

$$\frac{dx}{dt} = v_x \ , \qquad \frac{dy}{dt} = v_y$$

While we can also account for energy loss due to friction and collisions with the table boundaries later on, we can assume that the kinetic energy of the ball (and thus the magnitude of its velocity) remains constant. We also make the assumption that when the ball hits one of the table boundaries, that the collision is perfectly elastic and mirror-like; in other words the billiard's angle of incidence θ_i with the wall is equal to its angle of reflection θ_f:

$$\theta_i = \theta_f$$

Our task is then to plot the trajectory of the ball. While the trajectory is simple to plot between collisions (by Euler's method), some basic vector manipulation is required to determine the new velocity vector after the collision. If we suppose the initial velocity before a given collision can be denoted $\vec{v}_i = [(v_i)_x, (v_i)_x]$, then the components of the post-collision velocity vector $\vec{v}_f$ depend on which component of $\vec{v}_i$ was perpendicular and parallel to the table boundary. In short, given the perpendicular component $(v_i)_\perp$ and the parallel component $(v_i)_\parallel$, we get the components of the reflected velocity vector as:

$$(v_f)_\perp = -(v_i)_\perp \ , \quad (v_f)_\parallel = (v_i)_\parallel$$

While this is conceptually fairly straight-forward, the implementation is relatively difficult. A number of interesting difficulties arise, as I discuss in the next sections.

2. Numerical Methods

To solve the billiards problem numerically, we use Euler's method to solve the following ODE's from last section:

$$\frac{dx}{dt} = v_x\ , \qquad \frac{dy}{dt} = v_y$$

As we have seen many times in class, after providing some initial conditions, the motion of the ball at time step i in x and y is given by:

$$x_i = x_{i-1} + \frac{dx}{dt}\Delta t = x_{i-1} + (v_x)_{i-1}\Delta t$$
$$y_i = y_{i-1} + \frac{dy}{dt}\Delta t = y_{i-1} + (v_y)_{i-1}\Delta t$$

We can derive these two results by recalling that in order to estimate $\frac{dx}{dt}$ and $\frac{dy}{dt}$, we can use backwards differencing as follows for all indices $i \leq N$ for some number of points N:

$$\frac{dx}{dt} \approx \frac{x_i - x_{i-1}}{\Delta t}\ , \quad \frac{dy}{dt} \approx \frac{y_i - y_{i-1}}{\Delta t}$$

We then rearrange the two equations to obtain:

$$x_i = x_{i-1} + \frac{dx}{dt}\Delta t\ , \quad y_i = y_{i-1} + \frac{dy}{dt}\Delta t$$

To calculate the new velocity vector after a collision, no overly complex numerical methods are required.

3. Results

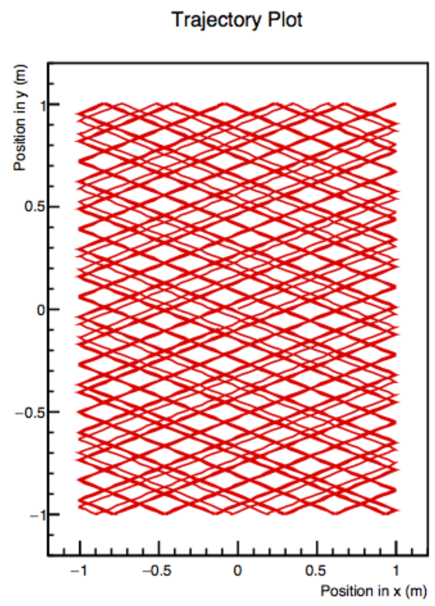
For my solution to the billiards problem, I decided to focus on having the most correct (bug-free) implementation, thus was only able to focus on the case of a square table. I considered the table to be a square region from $x = -1$ to $x = 1$ and $y = -1$ to $y = 1$. I chose to place the ball at the origin for the sake of simplicity, so $(x_0, y_0) = (0, 0)$ in all executions of my program. To allow the user to experiment with some different parameters, I left the choice of number of points, time precision, initial velocity and initial angle all the user. Default values are used if invalid parameters are selected, for example a negative number of points N.

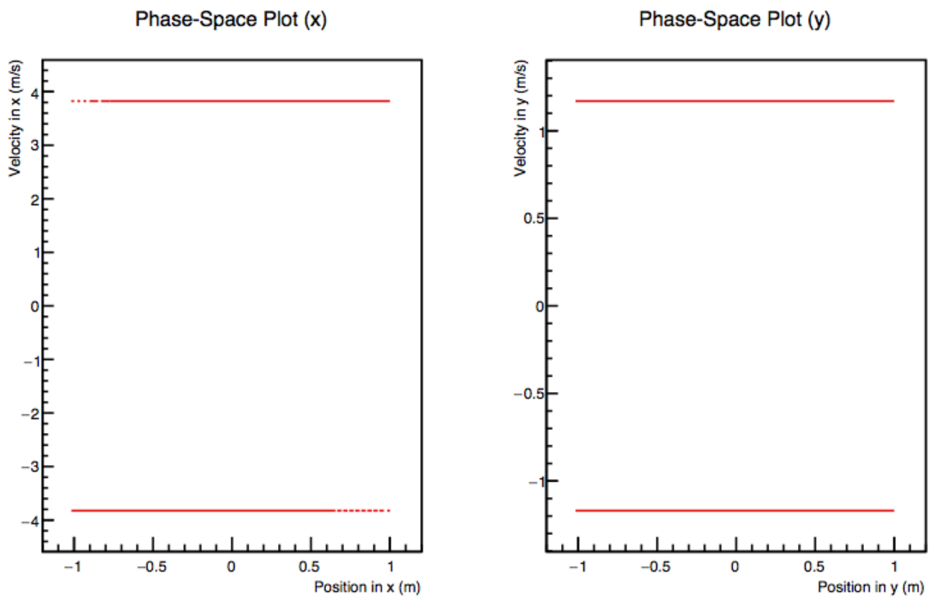
The major problem I encountered while implementing my solution was accurate collision detection, particularly around the corners. After writing my initial version of the program, I noticed that for large values of N and large time steps Δt, the billiard ball would eventually escape the corners of the table. The problem was largely caused by the following lines of code, which calculated the new velocity vector after a collision was detected:

```
if ((x[i] > xMax || x[i] < xMin) && (y[i] > yMax || y[i] < yMin)) {
        vx[i] = -vx[i-1];
        vy[i] = -vy[i-1];
}
else if (x[i] > xMax || x[i] < xMin) {
        vx[i] = -vx[i-1];
        vy[i] = vy[i-1];
}
else {
        vx[i] = vx[i-1];
        vy[i] = -vy[i-1];
}
```

While this algorithm does detect collisions in the corners of the table by simply changing the sign of both components of the velocity, it does not handle corner collisions effectively enough. The main issue with this approach is that it does not account for the following situation: suppose the billiard collides with the table boundaries in x and is very close to colliding in y. This algorithm adjusts the x component of the velocity accordingly, but on the next time step, the billiard may be off the table in the y direction. My solution solves this problem by detecting this exact scenario.

Overall, my implementation runs fairly well, but does not demonstrate the chaotic behavior seen in the "stadium" table configurations discussed in the textbook. For example, consider the following trajectory and phase-space plots generated by my program for $N = 10000$ points, a time step of $\Delta t = 0.01$, an initial velocity of 4.0 m/s, and an initial angle of $\theta = 17°$:

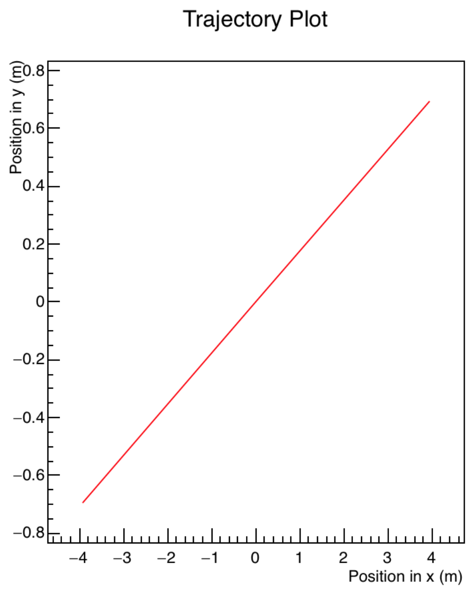




In future versions of my program, I would generalize the program to be able to handle different table geometries, like circular or "stadium" configurations in which chaos can be seen. It would also be interesting to allow the user to set the initial position of the billiard, which would be a minor change.

4. Stability of the Numerical Method

Overall, the program's behavior is stable given reasonable choices for Δt, the time step size, and the initial velocity vector. Time steps above $\Delta t = 2$ tend to be unreliable, given high velocities as well. Low-magnitude velocities and small time steps, however, help the program behave predictably and accurately. Problems that arise during executions of the program are not typically caused by approximation error in Euler's method; they tend to be caused by collisions around the corners. For example, given $N = 10000$, a time step size of $\Delta t = 20$, an initial velocity of $v = 20$ m/s, and an angle of $\theta = 10°$, we see that the billiard is able to escape out of both corners in the following example:


Trajectory Plot
Position in y (m)
Position in x (m)